

Self-similar Bianchi models I: Class A models

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Abstract

We present a study of Bianchi class A tilted cosmological models admitting a proper homothetic vector field together with the restrictions, both at the geometrical and dynamical level, imposed by the existence of the simply transitive similarity group. The general solution of the symmetry equations and the form of the homothetic vector field are given in terms of a set of arbitrary integration constants. We apply the geometrical results for tilted perfect fluids sources and give the general Bianchi II self-similar solution and the form of the similarity vector field. In addition we show that self-similar perfect fluid Bianchi VII₀ models and irrotational Bianchi VI₀ models do not exist.

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1 Introduction

The simplest generalisation of the Friedmann-Lemaître universes are the Spatially Homogeneous (SH) models which admit a local group G_r ($r = 3, 4$) of isometries acting on 3-dimensional spacelike hypersurfaces. From the physical point of view they serve as the basis in the study of anisotropies at early times of the universe evolution and play a key role in the understanding of the underlying geometrical/physical properties and insights of more general cosmological solutions of the Field Equations (FE):

$$R_{ab} = T_{ab} - \frac{T}{2}g_{ab}. \quad (1.1)$$

An extensively studied subclass of SH models are the Bianchi models where $r = 3$ which can be divided further into two major subclasses: orthogonal Bianchi cosmologies where the fluid four-velocity n^a is orthogonal to the group orbits [1] leading to the *non-tilted* models and Bianchi cosmologies where the normalised fluid velocity u^a is not aligned with n^a representing *tilted* models [2].

Although in SH models, the FE are reduced to a system of ordinary differential equations, not many exact solutions are known, especially in the case $r = 3$. This has initiated the examination of their dynamics using more sophisticated and predictive methods: the FE are reformulated as an autonomous system and using the methods of the theory of dynamical systems, the evolution of a specific model is studied in the so called *state space* which represents the set of all the physical states (at some instant of time) of the corresponding model [3, 4]. The main idea is to identify the evolution of a specific model with an orbit (solution curve) in state space and examine its geometrical-with direct physical interpretation-properties. One of the striking results of this approach is that *transitively self-similar* SH models are important in describing the asymptotic (i.e. at early and late times) states of the evolution for more general SH models. In particular, self-similar SH models arise as equilibrium points which determine various stable and unstable invariant submanifolds of the state space of the evolution equations. The determination of these invariant submanifolds allows to gain deeper insight into the intermediate evolution of the SH cosmological models. Furthermore there was

a strong evidence that any SH model could be approximated by self-similar SH models in the asymptotic regimes i.e. near the initial cosmological singularity and at late times. However it has been proved recently [5, 6] that Bianchi VIII and IX models are not (asymptotically) self-similar providing a counterexample in this conjecture. Nevertheless Bianchi type IX cosmologies are successively approximated (at the asymptotic regimes) by an infinite sequence of self-similar models (Kasner vacuum solutions) reinforce the belief that self-similar models have a significant role in the structure of the SH dynamical state space.

It becomes clear that self-similar Bianchi models are of primary cosmological importance. We will focus attention on Bianchi models ($r = 3$) admitting a proper homothetic vector field (HVF) \mathbf{H} acting *simply transitively* on space-time:

$$\mathcal{L}_{\mathbf{H}}g_{ab} = 2\psi g_{ab} \quad (1.2)$$

where $\psi = \text{const.}$ is the homothetic factor which essentially represents the (constant) scale transformation of the geometrical and dynamical variables.

The main goal of this paper is to provide the complete set of self-similar Bianchi models of class A (in particular for the "missing" types II, VI₀, VII₀) *irrespective* the matter source of the gravitational field. In the case of non-vacuum fluid models the four-velocity subjects only to the assumption that is mapped conformally by the 4-dimensional similarity group. We utilize the purely geometrical results and study the case of tilted perfect fluid models. Due to the intrinsic complexity of the tilted models, up to date, not all the self-similar models have been determined adding to the difficulty of qualitatively analysing Bianchi models. Therefore the self-similar metrics presenting in this paper can be used to determine the general self-similar tilted perfect fluid Bianchi models of class A (note that in the vacuum and non-tilted case considerable information are available [7]).

An outline of the paper is as follows: section 2 is devoted to a brief presentation of some of the basic properties of Bianchi models using the metric approach in which the basic variables are the frame components of the metric w.r.t. the canonical 1-forms [8, 9]. We then introduce a 1+3 decomposition of the tilted fluid velocity u^a and show that the frame components of u^a satisfy restrictions imposed by the structure of the isometry algebra of the SH models. In section 3 we confine our study to Bianchi class A models and present a set of constraints coming from the existence of a proper HVF. These constraints are used to determine explicitly the frame components of the metric, the fluid velocity and the corresponding similarity vector fields. In section 4 we discuss the physical implications of these, purely geometrical, results and examine the existence of tilted perfect fluid models. For Bianchi II models we regain the *general* self-similar tilted perfect fluid solution and prove that *there are no irrotational self-similar tilted Bianchi VI₀ models*. We also show that *there no self-similar tilted perfect fluid Bianchi VII₀ models*. We conclude in section 5 discussing our results and further avenues of research.

2 Preliminaries

In this section we briefly review some basic features of the Bianchi models regarding the effects, both at the level of Geometry and Physics, which arise from the existence of the 3-parameter group of isometries. Since the theory is well known (see e.g. [3] and [10] and references cited therein) we present here only those results which will be useful in the subsequent sections. Following Ellis et al. [3] we use the so called *metric approach* in which the basic variables are the frame components of the metric with respect to the set of canonical 1-forms. Although alternative techniques provide powerful unified formalisms of qualitatively analysing Bianchi models, the metric approach has the advantage of dealing with coordinate components hence we can determine the explicit (local) form of the self-similar metrics and consequently to study the resulting models directly. Throughout the paper the index convention is such that latin indices a, b, c, \dots are tensor components and take the values 0, 1, 2, 3 whereas Greek indices $\alpha, \beta, \gamma, \dots = 1, 2, 3$ denotes frame components (non-tensorial) of the corresponding quantities. A semicolon (;) denotes covariant differentiation

and a comma (,) denotes partial differentiation w.r.t. following index coordinate.

2.1 Geometry

Bianchi models are characterized by the existence of a G_3 Lie group of isometries acting on 3-dimensional spacelike orbits \mathcal{C} with generators the Killing vector fields (KVF) \mathbf{X}_α tangent to the group orbits. The spacelike hypersurfaces \mathcal{C} and the associated group G_3 of isometries uniquely determine a unit timelike congruence n^a ($n^a n_a = -1$) normal to the spatial foliations \mathcal{C} which satisfies:

$$n_{[a;b]} = 0 = n_{a;b} n^b \Leftrightarrow \frac{1}{2} \mathcal{L}_n g_{ab} \equiv n_{a;b} = \sigma_{ab} + \frac{\theta}{3} h_{ab} \quad (2.1)$$

where $\sigma_{ab}, \theta = n_{a;b} g^{ab}, h_{ab} = g_{ab} + n_a n_b$ are the trace-free symmetric shear tensor, the expansion and the projection tensor respectively associated with the timelike congruence n^a [11]. It turns out that the (irrotational and geodesic) timelike congruence n^a is invariant under the action of the group G_3 i.e. the KVF's commute with n^a :

$$[\mathbf{X}_\alpha, \mathbf{n}] = 0. \quad (2.2)$$

Therefore local coordinates $\{t, x^\alpha\}$ can be chosen such that $n_a = -t_{,a}$ and the spacelike foliations \mathcal{C} coincide with the hypersurfaces of constant t [1].

Introducing a set of invariant basis of vector fields \mathbf{e}_α and its dual ω^β such that:

$$[\mathbf{X}_\alpha, \mathbf{e}_\beta] = 0 \quad [\mathbf{X}_\alpha, \omega^\beta] = 0 \quad (2.3)$$

the full four dimensional metric can be written:

$$ds^2 = -dt^2 + g_{\alpha\beta}(t) \omega^\alpha \omega^\beta \quad (2.4)$$

and we have ensured the constancy of the 3-metric $g_{\alpha\beta}$ on each orbit \mathcal{C} .

In addition the invariant basis \mathbf{e}_α and its dual ω^α satisfy the corresponding commutation relations:

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = -C_{\alpha\beta}^\gamma \mathbf{e}_\gamma, \quad d\omega^\gamma = \frac{1}{2} C_{\alpha\beta}^\gamma \omega^\alpha \wedge \omega^\beta \quad (2.5)$$

where d stands for the usual exterior derivative of 1-forms, \wedge denotes the usual exterior product¹ and $C_{\alpha\beta}^\gamma = -C_{\beta\alpha}^\gamma$ are the structure constants of the group.

2.2 Physics

As we have mentioned in the Introduction, attention is focused on perfect fluid sources which can be described in terms of the normalised four-velocity of the fluid u^a ($u^a u_a = -1$):

$$T_{ab} = (\tilde{\mu} + \tilde{p}) u_a u_b + \tilde{p} g_{ab} \quad (2.6)$$

where $\tilde{\mu}, \tilde{p}$ are the energy density and the pressure measured by the observers comoving with the fluid velocity u^a . Using the FE we may express the Ricci tensor in a similar form:

$$R_{ab} = (\tilde{\mu} + \tilde{p}) u_a u_b + \frac{(\tilde{\mu} - \tilde{p})}{2} g_{ab}. \quad (2.7)$$

¹Note that the sign in (2.5) depends on the choice of the initial condition of the differential equations (2.3).

A straightforward consequence of the spatial homogeneity is that there is only one essential dynamical coordinate t . This follows from the invariance of the matter tensor under the action of the G_3 [16]:

$$\mathcal{L}_{\mathbf{X}_\alpha} T_{ab} = \mathcal{L}_{\mathbf{X}_\alpha} R_{ab} = 0, [\mathbf{X}_\alpha, \mathbf{u}] = 0, \mathbf{X}_\alpha(\tilde{\mu}) = \mathbf{X}_\alpha(\tilde{p}) = 0. \quad (2.8)$$

We note that the last two equations imply that perfect fluid models necessarily have a barotropic equation of state $\tilde{p} = \tilde{p}(\tilde{\mu})$. Furthermore and for the purposes of the present paper, it is convenient to decompose u^a parallel and perpendicular to n^a as follows:

$$\mathbf{u} = \Gamma \mathbf{n} + \Gamma \Delta_\alpha \omega^\alpha = \Gamma \mathbf{n} + \Gamma B^\alpha \mathbf{e}_\alpha \quad (2.9)$$

where $B^\alpha(t), \Delta_\alpha(t)$ are the frame components of the four-velocity u^a and Γ is a smooth function of the time coordinate t satisfying the constraint:

$$\Gamma = (1 - B^\alpha \Delta_\alpha)^{-\frac{1}{2}} \quad (2.10)$$

and $B^\alpha \Delta_\alpha < 1$.

The fluid velocity u^a , although invariant under the action of the group of isometries, *is not* necessarily parallel to the normal timelike congruence n^a , leading to *tilted* perfect fluid models i.e. the fluid flow lines are not orthogonal to the surfaces of spatial homogeneity (the quantity $B^\alpha \Delta_\alpha$ of the projection of u^a is directly connected with the so called "tilt direction" [2]). In order to establish the relation between the dynamical quantities defined by the timelike vector fields n^a, u^a , we consider the 1+3 decomposition of the matter tensor induced by the normal timelike vector field n^a [11]:

$$T_{ab} = \mu n_a n_b + p h_{ab} + 2q_{(a} n_{b)} + \pi_{ab} \quad (2.11)$$

where

$$\mu = T_{ab} n^a n^b, \quad p = \frac{1}{3} T_{ab} (n^a n^b + g^{ab}), \quad q_a = -h_a^c T_{cd} n^d, \quad \pi_{ab} = h_a^c h_b^d T_{cd} - \frac{1}{3} (h^{cd} T_{cd}) h_{ab} \quad (2.12)$$

are the dynamical quantities relative to n^a .

Using equations (2.6) and (2.11) we obtain the relations [12]:

$$\mu = \tilde{\mu} + \Gamma^2 v^2 (\tilde{\mu} + \tilde{p}) \quad , \quad p = \tilde{p} + \frac{1}{3} \Gamma^2 v^2 (\tilde{\mu} + \tilde{p}) \quad (2.13)$$

$$q_a = \Gamma^2 (\tilde{\mu} + \tilde{p}) Y_a \quad , \quad \pi_{ab} = \Gamma^2 (\tilde{\mu} + \tilde{p}) \left[Y_a Y_b - \frac{1}{3} Y^2 (g_{ab} + n_a n_b) \right] \quad (2.14)$$

where

$$\mathbf{Y} = B^\alpha \mathbf{e}_\alpha = \Delta_\alpha \omega^\alpha \quad , \quad Y^2 = Y^a Y_a = B^\alpha \Delta_\alpha. \quad (2.15)$$

Equations (2.14) indicate that the energy flux vector field q_a and the trace-free anisotropic pressure tensor π_{ab} are completely determined by the energy density and the pressure of the fluid and the spatial projection Y^a of u^a to the surfaces of spatial homogeneity.

The system of equations is supplemented with the conservation equation:

$$T_{;b}^{ab} = 0 \quad (2.16)$$

which for the case of the tilted perfect fluid source is reduced to:

$$\tilde{\mu}_{;a} u^a = -(\tilde{\mu} + \tilde{p}) \tilde{\theta} \quad (2.17)$$

$$\tilde{h}_a^k \tilde{p}_{;k} = -(\tilde{\mu} + \tilde{p}) \dot{u}_a \quad (2.18)$$

and we have used the 1+3 decomposition of the first derivatives of the four-velocity [11]:

$$u_{a;b} = \tilde{\sigma}_{ab} + \frac{\tilde{\theta}}{3}\tilde{h}_{ab} + \tilde{\Omega}_{ab} - \dot{u}_{[a}u_{b]} \quad (2.19)$$

where $\tilde{\sigma}_{ab}$, $\tilde{\theta} = u_{a;b}g^{ab}$, $\tilde{\Omega}_{ab}$, \dot{u}_a are the corresponding kinematical quantities of the fluid velocity u^a , namely, the trace-free shear tensor, expansion, rotation tensor and acceleration respectively and \tilde{h}_{ab} is the projection tensor associated with u^a . From (2.19) we obtain easily:

$$u_{[a;b]} = \tilde{\Omega}_{ab} - \dot{u}_{[a}u_{b]}. \quad (2.20)$$

$$\frac{1}{2}\mathcal{L}_{\mathbf{u}}g_{ab} \equiv u_{(a;b)} = \tilde{\sigma}_{ab} + \frac{\tilde{\theta}}{3}\tilde{h}_{ab} - \dot{u}_{(a}u_{b)}. \quad (2.21)$$

Due to spatial homogeneity (third and fourth of equations (2.8)) all the (kinematical and dynamical) quantities are invariant under the G_3 (which is equivalent with the vanishing of their Lie derivative with respect to the KVF's).

The decomposition (2.9) implies:

$$u_{a;b} = \Gamma_{,b}n_a + \Gamma n_{a;b} + \Gamma_{,b}\Delta_\alpha\omega_a^\alpha + \Gamma(\Delta_\alpha)_{,b}\omega_a^\alpha + \Gamma\Delta_\alpha\omega_{a;b}^\alpha. \quad (2.22)$$

We note that a similar relation holds if we apply the decomposition with respect to the invariant basis \mathbf{e}_α . The skew symmetric part of equation (2.22) gives:

$$u_{[a;b]} = -(\Gamma\Delta_\alpha)^\bullet n_{[b}\omega_{a]}^\alpha + \Gamma\Delta_\alpha\omega_{[a;b]}^\alpha \quad (2.23)$$

where a dot denotes differentiation w.r.t. t .

Contracting equation (2.23) with $e_\alpha^a e_\beta^b$ and using (2.5) we obtain:

$$u_{[a;b]}e_\alpha^a e_\beta^b = \frac{1}{2}\Gamma C_{\alpha\beta}^\gamma \Delta_\gamma. \quad (2.24)$$

In the case of a linear equation of state $\tilde{p} = (\gamma - 1)\tilde{\mu}$ the conservation law (2.18) and equation (2.20) give:

$$u_{[a;b]}e_\alpha^a e_\beta^b = \tilde{\Omega}_{ab}e_\alpha^a e_\beta^b \quad (2.25)$$

$$\dot{u}_{[a}u_{b]}e_\alpha^a e_\beta^b = 0. \quad (2.26)$$

Equation (2.24) relates the frame (spatial) components of $u_{[a;b]}$ and $\tilde{\Omega}_{ab}$ with the frame components Δ_α of the four-velocity u^a and the structure constants of the Lie group G_3 (we note that a rather similar equation has been proved in [2] based on the Ellis's orthonormal tetrad formalism). Therefore a simple inspection of the group structure of the spatial homogeneous models, jointly with the FE, show that the kinematical and dynamical variables of the Bianchi models depend directly on the group structure. For example it is straightforward to prove that there are no tilted perfect fluid Bianchi I models whereas Bianchi types VIII and IX must necessarily have non-zero vorticity [2]. Indeed in type I the structure constants all vanish thus $\tilde{\Omega}_{ab} = 0 \Leftrightarrow u_{[a;b]} = -\dot{u}_{[a}u_{b]}$ which, by means of equation (2.23) and the evolution equation of $\tilde{\Omega}_{ab}$ implies $\dot{u}_a n^a = 0$. Then, provided that $\tilde{\mu}_{;a} \neq 0$, the conservation law (2.18) leads to $\dot{u}_a = 0$ and the fluid is non tilted. In types VIII and IX, assuming that $\tilde{\Omega}_{ab} = 0$ equation (2.24) implies that $\Delta_\gamma = 0$ and the fluid is also non-tilted (and may be interpreted as anisotropic fluid). In the next section we draw attention only on Bianchi models of class A which possess a simply transitively self-similarity group. The case of class B will be studied in a forthcoming paper [13].

3 Self-similar Bianchi models of class A

A transitively self-similar Bianchi model admits a proper HVF \mathbf{H} which together with the isometry group, form a simply transitive group of homotheties H_4 acting on four dimensional orbits and has generators $\{\mathbf{H}, \mathbf{X}_\alpha\}$ spanned the associated Lie algebra denoted as \mathcal{H}_4 . Our main concern in this section will be to find the general solution of the symmetry equations (1.2) in Bianchi class A models. However it is useful to present first the restrictions on the kinematical and dynamical variables of the Bianchi models induced from the existence of the HVF. For this reason we shall need the following well known result [14]:

Proposition 1

For *every* unit timelike vector field u^a and a HVF $H^a = M_{\mathbf{H}}u^a + w^a$ where $w^a u_a = 0$ the following identity holds:

$$L_{\mathbf{H}}u_a = \left[(M_{\mathbf{H}})_{;k} u^k + \dot{u}_k H^k \right] u_k + 2\omega_{ab}H^b + M_{\mathbf{H}} \left[\dot{u}_a - (\ln M_{\mathbf{H}})_{;k} h_a^k \right] \quad (3.1)$$

where $\dot{u}_a, \omega_{ab}, h_a^k$ are the acceleration, the rotation and the projection tensor associated with the unit timelike vector field u^a .

3.1 Geometry

A direct calculation shows that the commutator of a HVF with a KVF is always a KVF which implies that the structure constants C_{ab}^d of the homothetic Lie algebra \mathcal{H}_4 satisfy $C_{ab}^4 = 0$. Moreover the Jacobi identities for C_{ab}^d are equivalent to the relations:

$$C_{\delta[a}^\gamma C_{bc]}^\delta = 0. \quad (3.2)$$

Following the considerations of the previous section, we decompose the HVF parallel and normal to n^a :

$$\mathbf{H} = H\mathbf{n} + A^\alpha \mathbf{X}_\alpha \quad (3.3)$$

where H and A^α are smooth functions of the space-time manifold.

From the commutator relations $[\mathbf{X}_\alpha, \mathbf{H}] = C_{\alpha 4}^\gamma \mathbf{X}_\gamma$ we get:

$$\mathbf{X}_\alpha(H) = 0 \quad , \quad \mathbf{n}(H) = \psi \quad (3.4)$$

$$\mathbf{n}(A^\beta) = 0 \quad , \quad \mathbf{X}_\alpha(A^\gamma) + A^\beta C_{\alpha\beta}^\gamma = C_{\alpha 4}^\gamma \quad (3.5)$$

where we have used (3.1) and the fact that, since \mathbf{n} is geodesic and irrotational, one has $[\mathbf{H}, \mathbf{n}] = -\mathbf{n}(H)\mathbf{n} + [A^\alpha \mathbf{X}_\alpha, \mathbf{n}] = -\psi\mathbf{n}$ where $\mathbf{n}(H) = \psi$. Equations (3.4) and (3.5) determine the functions A^γ up to constants of integration. However some of them can be ignored by means of appropriate transformations which maintain the form of the metric and the KVFs. Therefore the HVF is easily determined, provided that $C_{\alpha 4}^\gamma$ and $C_{\alpha\beta}^\gamma$ are given and subsequently, solving the symmetry equations (1.2), we determine the frame components $g_{\alpha\beta}(t)$ of the metric.

3.2 Physics

Using the well known relations $\mathcal{L}_{\mathbf{H}}R_{ab} = \mathcal{L}_{\mathbf{H}}T_{ab} = 0$ [15], it can be shown [16, 17] that the fluid flow lines are mapped conformally by the HVF and the equation of state is necessarily linear [18]. Moreover we deduce that in a self-similar SH perfect fluid model the "tilt angle" is constant and the projection Y^a of the fluid velocity is also conformally mapped by the HVF:

$$\mathbf{H}(\Gamma) = 0 \Leftrightarrow \Gamma_{;a} = 0 \quad , \quad \mathcal{L}_{\mathbf{H}}Y^a = -\psi Y^a \quad , \quad \mathcal{L}_{\mathbf{H}}Y_a = \psi Y_a \quad (3.6)$$

which by means of equations (2.22) and (2.23) imply:

$$u_{a;b} = \Gamma n_{a;b} + \Gamma (\Delta_\alpha)_{,b} \omega_a^\alpha + \Gamma \Delta_\alpha \omega_{a;b}^\alpha \quad (3.7)$$

$$u_{[a;b]} = -\Gamma \dot{\Delta}_\alpha n_{[b} \omega_{a]}^\alpha + \Gamma \Delta_\alpha \omega_{[a;b]}^\alpha. \quad (3.8)$$

We may further observe that the frame components B^α, Δ_α satisfy:

$$B^\alpha \Delta_\alpha = \text{const.} \quad (3.9)$$

Equations (2.8) and (3.6) can be used to determine, up to a set arbitrary constants of integration, the form of the fluid velocity.

3.3 Solution of the symmetry equations

Solving the system of equations (1.2), (3.4), (3.5) and (3.6) we can find the explicit expressions of the self-similar metric (the *frame components* of the metric i.e. the functions $g_{\alpha\beta}(t)$ in (2.4)), the homothetic vector field and the tilted four-velocity u^a for Bianchi class A models (we ignore the cases of Bianchi types I, VIII and IX since no tilted perfect fluid solutions exist). These results hold for a generic type of matter provided that, for fluid models, the four-velocity satisfies $\mathcal{L}_H u^a = -\psi u^a$.

Type II

In local coordinates $\{t, x, y, z\}$ and following the notations of [8], the KVs $\{\mathbf{X}_\alpha\}$ and the canonical 1-forms $\{\omega^\alpha\}$ are:

$$\mathbf{X}_1 = \partial_y, \quad \mathbf{X}_2 = \partial_z, \quad \mathbf{X}_3 = \partial_x + z\partial_y. \quad (3.10)$$

$$\omega^1 = dy - xdz, \quad \omega^2 = dz, \quad \omega^3 = dx. \quad (3.11)$$

The non-vanishing structure constants are $C_{23}^1 = 1$ and the Jacobi identities (3.2) imply that the remaining non-vanishing structure constants $C_{\beta 4}^\alpha$ are [19, 20]:

$$C_{14}^1 = a, \quad C_{24}^2 = 1, \quad C_{34}^3 = a - 1 \quad (3.12)$$

$$C_{14}^1 = 2, \quad C_{24}^2 = 1, \quad C_{34}^2 = 1, \quad C_{34}^3 = 1 \quad (3.13)$$

$$C_{14}^1 = a, \quad C_{24}^3 = 1, \quad C_{34}^2 = -1, \quad C_{34}^3 = a \quad (a^2 < 4) \quad (3.14)$$

We consider subcases:

Case A₁

HVF

$$\mathbf{H} = \psi t \partial_t + (a - 1 + b) \partial_x + ay \partial_y + z \partial_z \quad (3.15)$$

Fluid velocity

$$\Delta_1 = v_1 t^{(\psi-a)/\psi}, \quad \Delta_2 = t^{(\psi-1)/\psi} \left(v_2 + \frac{v_1 b \ln t}{\psi} \right), \quad \Delta_3 = v_3 t^{(\psi+1-a)/\psi} \quad (3.16)$$

Metric

$$g_{\alpha\beta} = \begin{pmatrix} c_{11} t^{2(\psi-a)/\psi} & t^{(2\psi-a-1)/\psi} \left[c_{12} + \frac{bc_{11}(\ln t)}{\psi} \right] & c_{13} t^{(2\psi-2a+1)/\psi} \\ t^{(2\psi-a-1)/\psi} \left[c_{12} + \frac{bc_{11}(\ln t)}{\psi} \right] & t^{2(\psi-1)/\psi} \left[c_{22} + \frac{b^2 c_{11}(\ln t)^2}{\psi^2} + \frac{2bc_{12} \ln t}{\psi} \right] & t^{(2\psi-a)/\psi} \left[c_{23} + \frac{bc_{13} \ln t}{\psi} \right] \\ c_{13} t^{(2\psi-2a+1)/\psi} & t^{(2\psi-a)/\psi} \left[c_{23} + \frac{bc_{13}(\ln t)}{\psi} \right] & c_{33} t^{2(\psi-a+1)/\psi} \end{pmatrix} \quad (3.17)$$

where $b = 0$ when $a \neq 1$.

Case A₂
HVF

$$\mathbf{H} = \psi t \partial_t + x \partial_x + \frac{x^2 + 4y}{2} \partial_y + (x + z) \partial_z \quad (3.18)$$

Fluid velocity

$$\Delta_1 = v_1 t^{(\psi-2)/\psi}, \Delta_2 = v_2 t^{(\psi-1)/\psi}, \Delta_3 = t^{(\psi-1)/\psi} \left(v_3 - \frac{v_2}{\psi} \ln t \right) \quad (3.19)$$

Metric

$$g_{\alpha\beta} = \begin{pmatrix} c_{11} t^{2(\psi-2)/\psi} & c_{12} t^{(2\psi-3)/\psi} & t^{(2\psi-3)/\psi} \left(c_{13} - \frac{c_{12} \ln t}{\psi} \right) \\ c_{12} t^{(2\psi-3)/\psi} & c_{22} t^{2(\psi-1)/\psi} & t^{2(\psi-1)/\psi} \left[c_{23} - \frac{c_{22} \ln t}{\psi} \right] \\ t^{(2\psi-3)/\psi} \left(c_{13} - \frac{c_{12} \ln t}{\psi} \right) & t^{2(\psi-1)/\psi} \left[c_{23} - \frac{c_{22} \ln t}{\psi} \right] & t^{2(\psi-1)/\psi} \left[c_{33} + \frac{c_{22} (\ln t)^2}{\psi^2} - \frac{2c_{23} \ln t}{\psi} \right] \end{pmatrix} \quad (3.20)$$

Case A₃
HVF

$$\mathbf{H} = \frac{a}{2-p_2} t \partial_t + (ax + z) \partial_x + \left(ay + \frac{z^2 - x^2}{2} \right) \partial_y - x \partial_z \quad (3.21)$$

Fluid velocity

$$\Delta_1 = v_1 t^{p_2-1}, \Delta_2 = t^{p_2/2} \left[v_{23} \cos \left(\frac{p_1 \ln t}{2} \right) + v_{32} \sin \left(\frac{p_1 \ln t}{2} \right) \right], \Delta_3 = \frac{a}{2-p_2} \left[\Delta_2 - t (\Delta_2)_{,t} \right] \quad (3.22)$$

Metric

$$g_{\alpha\beta} = \begin{pmatrix} c_{11} t^{2(p_2-1)} & g_{21} & g_{31} \\ \frac{a}{2-p_2} \left[t (g_{13})_{,t} - 2(p_2-1) g_{13} \right] & \frac{a}{2-p_2} \left[t (g_{23})_{,t} - p_2 g_{23} + g_{33} \right] & g_{32} \\ t^{(3p_2-2)/2} \left[c_{13} \cos \left(\frac{p_1 \ln t}{2} \right) + c_{31} \sin \left(\frac{p_1 \ln t}{2} \right) \right] & \frac{a}{2(2-p_2)} \left[t (g_{33})_{,t} - 2(p_2-1) g_{33} \right] & g_{33} \end{pmatrix} \quad (3.23)$$

where:

$$p_1 = \frac{\sqrt{4-a^2}}{\psi}, p_2 = \frac{2\psi-a}{\psi} \quad (3.24)$$

$$g_{33} = t^{p_2} [c_{33} + c_{32} \cos(p_1 \ln t) + c_{23} \sin(p_1 \ln t)]. \quad (3.25)$$

We note that in all cases A₁, A₂, A₃ the quantities $v_\alpha, c_{\alpha\beta}, a, b$ are constants of integration.

Type VI₀

The KVs $\{X_\alpha\}$ and the dual basis $\{\omega^\alpha\}$ are:

$$\mathbf{X}_1 = \partial_y, \mathbf{X}_2 = \partial_z, \mathbf{X}_3 = \partial_x + y \partial_y - z \partial_z. \quad (3.26)$$

$$\omega^1 = e^{-x} dy, \omega^2 = e^x dz, \omega^3 = dx. \quad (3.27)$$

In this case the non-vanishing structure constants of the isometry group $C_{13}^1 = -C_{23}^2 = 1$ and the Jacobi identities (3.2) imply that the remaining non-vanishing structure constants $C_{\beta 4}^\alpha$ are:

$$C_{14}^1 = a, \quad C_{24}^2 = b \quad (3.28)$$

and the HVF assumes the form:

$$\mathbf{H} = \psi t \partial_t + D \partial_x + ay \partial_y + bz \partial_z. \quad (3.29)$$

Furthermore the four-velocity and the metric are:

Fluid velocity

$$\Delta_1 = v_1 t^{(\psi+D-a)/\psi}, \quad \Delta_2 = v_2 t^{(\psi-D-b)/\psi}, \quad \Delta_3 = v_3 t \quad (3.30)$$

Metric

$$g_{\alpha\beta} = \begin{pmatrix} c_{11} t^{2(\psi+D-a)/\psi} & c_{12} t^{(2\psi-a-b)/\psi} & c_{13} t^{(2\psi+D-a)/\psi} \\ c_{12} t^{(2\psi-a-b)/\psi} & c_{22} t^{2(\psi-D-b)/\psi} & c_{23} t^{(2\psi-D-b)/\psi} \\ c_{13} t^{(2\psi+D-a)/\psi} & c_{23} t^{(2\psi-D-b)/\psi} & c_{33} t^2 \end{pmatrix} \quad (3.31)$$

where $v_\alpha, c_{\alpha\beta}, a, b, D$ are constants. We note that when the constants c_{13}, c_{23} are non-vanishing we may use the freedom $y = \pm \tilde{y}/c_{13}$ and $z = \pm \tilde{z}/c_{23}$ to set $c_{13} = c_{23} = \pm 1$.

Type VII₀

The KVs $\{X_\alpha\}$ and the dual basis $\{\omega^\alpha\}$ are:

$$\mathbf{X}_1 = \partial_y, \quad \mathbf{X}_2 = \partial_z, \quad \mathbf{X}_3 = \partial_x - z \partial_y + y \partial_z. \quad (3.32)$$

$$\omega^1 = \cos x dy + \sin x dz, \quad \omega^2 = -\sin x dy + \cos x dz, \quad \omega^3 = dx. \quad (3.33)$$

The non-vanishing structure constants are $C_{13}^2 = -C_{23}^1 = 1$ and the remaining non-vanishing structure constants $C_{\beta 4}^\alpha$ are:

$$C_{14}^1 = C_{24}^2 = a. \quad (3.34)$$

The HVF has the form:

$$\mathbf{H} = \psi t \partial_t + D \partial_x + ay \partial_y + az \partial_z. \quad (3.35)$$

The frame components of the four-velocity and the metric are:

Fluid velocity

$$\begin{aligned} \Delta_1 &= t^{(\psi-a)/\psi} \left[v_1 \cos \left(\frac{D \ln t}{\psi} \right) + v_2 \sin \left(\frac{D \ln t}{\psi} \right) \right] \\ \Delta_2 &= t^{(\psi-a)/\psi} \left[-v_1 \sin \left(\frac{D \ln t}{\psi} \right) + v_2 \cos \left(\frac{D \ln t}{\psi} \right) \right] \\ \Delta_3 &= v_3 t \end{aligned} \quad (3.36)$$

Metric

$$g_{\alpha\beta} = \begin{pmatrix} t^{2(\psi-a)/\psi} \left[c_{21} \cos \left(\frac{2D \ln t}{\psi} \right) + c_{12} \sin \left(\frac{2D \ln t}{\psi} \right) + \frac{c_{11}}{2} \right] & g_{21} & g_{31} \\ t^{2(\psi-a)/\psi} \left[-c_{21} \sin \left(\frac{2D \ln t}{\psi} \right) + c_{12} \cos \left(\frac{2D \ln t}{\psi} \right) \right] & c_{12} t^{2(\psi-a)/\psi} - g_{11} & g_{32} \\ t^{2(\psi-a)/\psi} \left[c_{31} \cos \left(\frac{D \ln t}{\psi} \right) + c_{13} \sin \left(\frac{D \ln t}{\psi} \right) \right] & t^{(2\psi-a)/\psi} \left[-c_{31} \sin \left(\frac{D \ln t}{\psi} \right) + c_{13} \cos \left(\frac{D \ln t}{\psi} \right) \right] & c_{33} t^2 \end{pmatrix} \quad (3.37)$$

where $v_\alpha, c_{\alpha\beta}, a, D$ are constants.

Applying the translation $x = \tilde{x} + A$ the form of the metric and the KVs remain invariant. Under this freedom in the choice of the x -coordinate we may set one of the constants c_{13}, c_{31} , say c_{31} , equal to zero i.e. $c_{31} = 0$. The other arbitrary constant, if not vanish, is fixed so that $c_{13} = \pm 1$ using the constant scale transformations $y = \pm \tilde{y}/c_{13}$ and $z = \pm \tilde{z}/c_{13}$.

4 Tilted perfect fluid solutions

Having found the general form of the Bianchi type II, VI₀ and VII₀ metrics which admit a proper HVF, we may utilize these results to locate those space-times which have a tilted perfect fluid as their source. In this case the FE (1.1) reduced to a purely, although highly non-linear, algebraic system of equations. However using the restrictions on the frame components of the fluid velocity described in sections 2,3 we have been able to solve completely the system of equations for Bianchi type II and VII₀ models and present some useful results concerning the Bianchi VI₀ models.

The outline of the method is as follows:

For each self-similar Bianchi metric we compute the Ricci tensor R_{ab} and using the decomposition (2.7) we construct the tensor $R_{ab} - \gamma \tilde{\mu} u_a u_b - \frac{(2-\gamma)}{2} \tilde{\mu} g_{ab}$ which, together with (2.10), lead to 11 algebraic equations in 16 unknowns $c_{\alpha\beta}, a, b, \Gamma, v_\alpha, \gamma, \tilde{\mu}, D, \psi$. We note that the number of the essential parameters needed for the general solution is reduced considerable for each Bianchi model as we shall see in the analysis presented below.

Type II

Tilted perfect fluid Bianchi type II models has necessarily zero rotation [2]. By virtue of equation (2.25) it turns out that $\Delta_1 = 0$. Furthermore, taking into consideration (2.9), the frame component Δ_2 can be set so that $\Delta_2 = 0$. Therefore the *general form* of the tilted velocity in self-similar Bianchi II models is:

$$u_a = \Gamma (n_a + \Delta_3 \omega_a^3). \quad (4.1)$$

Analytical computations show that only in case A₁ a perfect fluid solution exist which subsequently is the most *general self-similar tilted perfect fluid Bianchi II solution*. The metric is given by (3.17) with $b = c_{13} = c_{23} = 0$, $c_{12} = 1$ and the remaining non-vanishing constants are:

$$c_{11} = \frac{2p(3-4p)}{5p-1}, \quad c_{22} = \frac{(1-4p)}{2p(5p-3)}, \quad c_{33} = \frac{(4p-3)(5p-3)}{(4p-1)(5p-1)} \quad (4.2)$$

where p is constant and we have set $\psi = \frac{1}{1-3p}$, $a = \frac{p-1}{3p-1}$.

The kinematical and dynamical quantities of this class of self-similar tilted perfect fluid models are:

$$\tilde{\mu} = \frac{2p(2p+1)}{t^2}, \quad \gamma = \frac{2}{2p+1}, \quad \Gamma = \frac{\sqrt{2}(3-4p)(5p-3)}{2\sqrt{(5p-3)(3-4p)(9p-4)}} \quad (4.3)$$

where without loss of generality $v_3 = 1$.

Taking into account the overall signature of the space-time, the positivity of the quantities (4.3) show that these models are defined for $p \in (\frac{3}{5}, \frac{3}{4})$ or $p \in (0, \frac{1}{5})$. The first set represents an exact Bianchi II perfect fluid solution where the parameter γ lies in the interval $[4/5, 10/11]$. However this solution is not physically acceptable because the magnitude of the shear tensor $\tilde{\sigma}_{ab}\tilde{\sigma}^{ab}$ and the measure of the acceleration \dot{u}^a are both negative. In the second set of solutions the state parameter $\gamma \in [10/7, 2]$ and has been found by Hewitt [21, 22].

Types VI₀, VII₀

In contrast with the Bianchi type II models where significant information are available, little work has been done for tilted perfect fluid Bianchi VI₀ and VII₀ models even in the special case of self-similar models,

due to additional degrees of freedom in the determination of the general solution. The problem is simplified with the help of equations (2.25) and (2.26) which, in conjunction with the FE and the generic form of the self-similar metrics (3.31) and (3.37), can be used to make some interesting comments concerning the known Bianchi VI₀ solutions. As far as we know the only known tilted perfect fluid solution is due to Rosquist and Jantzen [23, 24]. These Bianchi VI₀ models have non-zero vorticity and the parameter $\gamma \in (1.0411, 1.7169)$ ($\gamma \neq 10/9$). In the case of type VII₀ we show that *there are no non-vacuum self-similar solutions*.

From the above discussion one may wonder if there are non-rotating self-similar Bianchi VI₀ and VII₀ tilted perfect fluid models at all. Although the isometry group G_3 of Bianchi VI₀ and VII₀ models, shares the common property of all the Bianchi models (except types VIII and IX) to admit an Abelian G_2 subgroup, self-similar models are *necessarily rotating*. In fact we prove the following:

Proposition 2

There are no self-similar and irrotational tilted perfect fluid Bianchi type VI₀ and VII₀ models.

Proof

Projecting equations $\mathcal{L}_{\mathbf{X}_\alpha} u_a = 0$ and $\mathcal{L}_{\mathbf{H}} u_a = \psi u_a$ with \mathbf{H} and \mathbf{X}_α and subtracting we obtain:

$$2u_{[a;b]} H^a X_\alpha^b = -\psi u_k X_\alpha^k + u_k C_{\alpha 4}^\beta X_\beta^k \quad (4.4)$$

which relates the projections of $u_{[a;b]}$ in the directions of the HVF and the KVs, with the structure of the homothetic algebra. Moreover equations (2.17) and (2.18) imply:

$$\dot{u}_{[a} u_{b]} H^a X_\alpha^b = \psi \frac{\gamma - 1}{\gamma} u_k X_\alpha^k. \quad (4.5)$$

By means of equation (2.25), the condition $\tilde{\Omega}_{ab} = 0$ implies that the frame components Δ_2 and Δ_3 vanish therefore the KVs $\mathbf{X}_1, \mathbf{X}_2$ are normal to the tilted velocity i.e. $X_1^a u_a = X_2^a u_a = 0$. Since $u_{[a;b]} = -\dot{u}_{[a} u_{b]}$ from (4.4) and (4.5) it follows that $\gamma = 2$ (we recall that $C_{34}^\alpha = 0$) i.e. the only transitively self-similar Bianchi VI₀ and VII₀ tilted perfect fluid models contain stiff fluid as its source. Then the FE (2.7), imply that the Ricci tensor has two eigenvectors X_1^a, X_2^a with zero eigenvalue:

$$R_{ab} X_1^b = R_{ab} X_2^b = 0. \quad (4.6)$$

Hence the KVs $\mathbf{X}_1, \mathbf{X}_2$ must be hypersurface orthogonal [25] and the metrics are necessarily diagonal. Taking the metrics (3.31) and (3.37) to be diagonal i.e. $c_{12} = c_{21} = c_{13} = c_{31} = c_{23} = c_{32} = 0$ a self-similar solution occurs only in type VI₀ (the type VII₀ solution corresponds to the flat Friedmann-Lemaître model). This solution is found to be unphysical ($\tilde{\mu} < 0$) in agreement with the general result that the *only non-vacuum self-similar Bianchi model with an Abelian subgroup of isometries acting orthogonally transitively is given by the metric (3.17) and (4.2)* [21].

In the case of rotating models our analysis shown that when $\gamma = 2$ the resulting type VI₀ model is unphysical ($\tilde{\mu} < 0$) hence there are no, physically acceptable, self-similar stiff fluid Bianchi VI₀ solutions. We have been unable to solve the resulting system of algebraic equations in full generality. However, although the family of solutions found in [24] belongs to the special classes satisfying the constraint $n_\alpha^\alpha = 0$, we expect that this family represents the most *general non-vacuum Bianchi VI₀ model*. In fact setting $a = (p_1 + p_2)\psi + 4\psi - b$ and $D = (2 - p_2) - b$ in (3.31) (p_1, p_2 are constants) it follows that the number of real degrees of freedom in a self-similar models is one (eleven equations and twelve unknowns). This issue is currently under investigation.

Concerning the type VII₀ models, the method described earlier in this section does not apply due to the complexity of the expression of the Ricci tensor. An alternative method to overcome the difficulties of computing the Ricci tensor, is the use of the Ricci identities for the timelike vector field n^a combined with equation (2.26) and has as follows:

The first set of equations we will need is the constraint equations for the timelike congruence n^a follow from the Ricci identities $R_{akbc}n^k = 2n_{a[c}b]$ (equation (72) of [11]):

$$h_c^a h_d^b (\sigma^{cd})_{;b} + q^a = 0. \quad (4.7)$$

Contracting (4.7) with the KVs X_α^a and using the first of (2.14) we obtain:

$$(\sigma_d^a X_\alpha^d)_{;a} + \Gamma \gamma \tilde{\mu} u_a X_\alpha^a = 0. \quad (4.8)$$

In addition expressing equation (2.26) in terms of the KVs² we get:

$$\dot{u}_{[a} u_{b]} X_\alpha^a X_\beta^b = 0. \quad (4.9)$$

The set of equations (4.8) and (4.9) is used to examine the consistency of the self-similar metrics (3.37) with the assumption of a tilted perfect fluid source. Because the expressions of the KVs and tilted velocity are given by (3.32) and (3.36) respectively, we compute the quantity $\sigma_d^a X_\alpha^d$ and the acceleration \dot{u}_a of the fluid. Replacing back to (4.8) and (4.9) we are left with a system of 6 algebraic equations whose solution gives the required answer. It is found that for every choice of the parameters, the energy density is always negative except in the case where the space-time reduced to the flat Robertson-Walker space-time. Hence *there are no self-similar and tilted perfect fluid Bianchi type VII₀ model except its flat RW counterpart.*

5 Discussion

In this paper we have been able to give the explicit forms of the non-vacuum self-similar Bianchi models of class A regardless the matter content of the gravitational field. Therefore the results found in section 3, although purely geometrical, can be used to determine the general tilted perfect fluid self-similar solution since the only limited kinematical assumption we have made is the conformally mapping of the fluid velocity i.e. $\mathcal{L}_{\mathbf{H}} u^a = -\psi u^a$. It follows that fluid models (non-tilted or tilted) with arbitrary energy-momentum tensor which admit a proper HVF, must included to the general models presented here.

An important aspect of the analysis given in sections 3 and 4 (which is valid for the entire class of Bianchi models) is the direct dependency of the vorticity on the structure of the similarity group. This observation enables us to make some straightforward comments regarding the behaviour of the non-vacuum Bianchi models class B. In particular from equations (2.25), (4.4) and (4.5) we deduce that for *all* irrotational Bianchi models, except type III, the fluid velocity frame components $\Delta_1 = \Delta_2 = 0$. Thus we expect that self-similar class B models subject to similar restrictions as in class A models. In fact it can be shown that self-similar and irrotational Bianchi IV and V models are only those for which $\gamma = 2$ [13]. It turns out that the corresponding KVs $\mathbf{X}_1, \mathbf{X}_2$ are also hypersurface orthogonal and the metric can be written in diagonal form in which case neither solution is physically acceptable hence *there are no Bianchi IV and V self-similar tilted perfect fluid solutions with zero vorticity.*

Concerning the rotational class A models one must to solve the high non-linear algebraic equations in order to answer definitely whether or not a self-similar perfect fluid model exists. We have been able to work efficiently on the case $\gamma = 2$ in Bianchi type VI₀ and show that this class of models, found by Wainwright et al. [26], do not admit a proper HVF. In addition based on the fact that there is only one essential parameter together with a rigorous, but not conclusive, study of the self-similar-metrics (3.31), we regard the assumption that the family of Rosquist-Jantzen solutions is the most general in type VI₀, as highly plausible.

²We can always write the invariant vector fields as combination of the KVs in the form $\mathbf{e}_\alpha = Z_\alpha^\beta \mathbf{X}_\beta$ where Z_α^β are time-independent smooth functions.

In the case of Bianchi type VII₀ our analysis shows that *self-similar models do not admit a perfect fluid interpretation*. As we have mentioned in the Introduction types VIII and IX also do not admit a perfect fluid interpretation and it is well known that the corresponding non-tilted models share the same feature (see e.g. Table 1 of [7]). These results indicate that, in connection with the *self-similarity breaking at late times* in Bianchi VII₀ non-tilted models [5, 27], it may be possible to relate the existence of a self-similar Bianchi model with the asymptotic self-similarity property of more general models. Consequently, it is an interesting and important task to determine under what conditions a *full self-similarity breaking* (i.e. in the asymptotic regimes) occurs and prove the conjecture that a *general perfect fluid Bianchi model is asymptotically self-similar at late times or near to the initial singularity if and only if a self-similar perfect fluid model exists*.

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